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The Numbers of Faces of Polytope Pairs and Unbounded Polyhedra

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Klee in 1966 proved that every simple d-polyhedron P with ν facets has at least $\nu-d+1$ vertices. Grünbaum speculated whether this result might be improved upon if one specified both the number of bounded and of unbounded facets of P. In 1974 Klee approached problems of this form from the point of view of pairs of simple polytopes while investigating the efficiency of a proposed algorithm to enumerate the vertices of a simple polytope defined by linear inequalities. In this paper we examine polytopes in a dual fashion to that of Klee, and strengthen and extend some of his results. Specifically, let P be a simplicial d-polytope with ν vertices and $\Sigma(P)$ be the simplicial $(d-1)$-complex associated with the boundary of P. Suppose, for a given vertex v of P, that we know the numbers of faces of various dimensions of $\mathrm{lk}_{\Sigma(P)}v$. Then we are able to determine tight upper and lower bounds for the possible numbers of faces of all dimensions of P and of $\Sigma(P)\backslash v$. As a consequence we resolve some open questions of Klee and settle a conjecture of Björner.

1. INTRODUCTION

A (*convex*) *polyhedron* is a non-empty intersection of a finite number of closed half-spaces in $\mathbf{R}^n$. A (*convex*) *polytope* is a bounded polyhedron; equivalently, it is the convex hull of a finite non-empty set of points in $\mathbf{R}^n$. We say P is a *d-polyhedron* (respectively, *d-polytope*) if P is a d-dimensional polyhedron (respectively, polytope).

Faces of a d-polyhedron P of dimension $0, 1, d-2$ and $d-1$ will be called *vertices*, *edges*, *ridges* (or *subfacets*) and *facets* of P, respectively, and the set of vertices of P will be denoted $V(P)$. For integer $0\leq j\leq d-1$ let $f_j(P)$ be the number of j-faces (j-dimensional faces) of P. The d-vector $f(P)=(f_0(P), f_1(P), \ldots, f_{d-1}(P))$ will be called the *f-vector* of P.

We will implicitly assume that any polyhedron P under consideration is pointed, i.e. $V(P)\neq\varnothing$. This is not a strong restriction because for every non-pointed polyhedron there exists a pointed polyhedron with isomorphic face lattice.

A d-polyhedron P is *simple* if every vertex of P is contained in exactly d facets of P. A d-polytope P is *simplicial* if every facet of P contains exactly d vertices of P. For every simplicial (respectively, simple) d-polytope P there exists a simple (respectively, simplicial) d-polytope P^* that is dual to P in the sense that there is an inclusion-reversing bijection between the set of faces of P and the set of faces of P^*, matching j-faces of P with $(d-j-1)$-faces of P^*, $-1\leq j\leq d$. Following [7] we let $\mathscr{P}_s^d$ be the set of all simplicial d-polytopes and define

$$f(\mathscr{P}_s^d)=\{f(P)\colon P\in\mathscr{P}_s^d\}.$$

Four interesting f-vector problems are:

(1) Determine the extremal values of $f_j(P)$, $0\leq j\leq d-2$, over the set of all simple d-polytopes P with ν facets. Equivalently, determine the extremal values of $f_j(P)$, $1\leq j\leq d-1$, over the set of all simplicial d-polytopes P with ν vertices.

(2) Characterize $\{f(P)\colon P$ is a simple d-polytope$\}$. Equivalently, characterize $f(\mathscr{P}_s^d)$.

(3) Determine the extremal values of $f_j(P)$, $0\leq j\leq d-2$, over the set of all unbounded simple d-polyhedra with ν facets, r of which are unbounded.

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(4) Characterize $\{f(P): P$ is an unbounded, simple d-polyhedron$\}$.

The first problem was solved by McMullen [14, 15] (the Upper Bound Theorem) and Barnette [1, 2] (the Lower Bound Theorem). McMullen's conjectured characterization of $f(\mathscr{P}_s^d)$ was verified by Billera and Lee [3, 4, 13] and Stanley [21]. Portions of problem (3) have been considered, e.g., by Klee [11] and Björner [5].

In this paper, which can be regarded as a sequel to [4], we will solve a strengthened form of the third problem (Theorem 4.2) and, within the context of simplicial complexes dual to simple polyhedra, will conjecture a characterization for problem (4) (Conjecture 5.1).

Throughout this paper we will adopt the convention that $\binom{-1}{0}=1$ and otherwise

$$\binom{a}{b}=0 \text{ if } a<b \text{ or } b<0.$$

2. Simplicial Complexes and h-vectors

A *simplicial complex* Δ on the finite set $V=V(\Delta)$ is a non-empty collection of subsets of V with the property that $\{v\}\in\Delta$ for all $v\in V$ and that $F\in\Delta$ whenever $F\subseteq G$ for some $G\in\Delta$. For $F\in\Delta$ we say F is a *face* of Δ and the *dimension* of F, dim F, equals j if card $F=j+1$. In this case we call F a *j-face* of Δ. The *dimension* of Δ, dim Δ, is defined to be $\max\{\dim F: F\in\Delta\}$. If $\dim\Delta=d-1$, we will refer to Δ as a simplicial $(d-1)$-complex. For simplicial $(d-1)$-complex Δ, faces of Δ of dimension $0, 1, d-2$ and $d-1$ will be called *vertices, edges, ridges* (or *subfacets*) and *facets* of Δ, respectively. As with polyhedra, we denote the number of j-faces of Δ by $f_j(\Delta)$, $0\le j\le d-1$, and define $f(\Delta)=(f_0(\Delta), f_1(\Delta),\ldots,f_{d-1}(\Delta))$ to be the *f-vector* of Δ. We also set $f_{-1}(\Delta)=1$ (because $\varnothing\in\Delta$) and we take $f_j(\Delta)=0$ if $j<-1$ or $j>d-1$.

Let $F=\{v_1,v_2,\ldots,v_k\}$ be a subset of V. The power set of F, $\{G\subseteq V: G\subseteq F\}$, will be denoted $\bar{F}$. For convenience, we may sometimes write $v_1v_2\cdots v_k$ for F and $\overline{v_1v_2\cdots v_k}$ for $\bar{F}$.

Two simplicial complexes Δ_1 and Δ_2 are *isomorphic*, denoted $\Delta_1\cong\Delta_2$, provided there is a bijection between $V(\Delta_1)$ and $V(\Delta_2)$ which induces a bijection between Δ_1 and Δ_2.

Suppose Δ is a simplicial complex and $F\in\Delta$. Define the *link* of F in Δ to be the simplicial complex

$$\mathrm{lk}_\Delta F=\{G\in\Delta: G\cap F=\varnothing, G\cup F\in\Delta\}.$$

If, further, we have $F\neq\varnothing$, the *deletion* of F from Δ is the simplicial complex

$$\Delta\backslash F=\{G\in\Delta: F\not\subseteq G\}.$$

By $|\Delta|$ is meant the underlying topological space of Δ. If $|\Delta|$ is a topological $(d-1)$-ball (respectively, sphere) we say Δ is a *simplicial $(d-1)$-ball* (respectively, *simplicial $(d-1)$-sphere*). For simplicial $(d-1)$-ball Δ, write $\partial\Delta$ for the simplicial $(d-2)$-sphere associated with $\partial|\Delta|$. The complex $\partial\Delta$ is called the *boundary* of Δ and it is known [17] that

$$\partial\Delta=\cup\{\bar{F}: F \text{ is a ridge of } \Delta \text{ contained in exactly one facet of } \Delta\}.$$

Each $F\in\partial\Delta$ will be called a *boundary face* of Δ; the remaining faces of Δ are its *interior faces* and the set of interior faces will be denoted Δ°.

We conclude this section with the notion of the h-vector of a simplicial complex, which experience has shown to have more algebraic significance than the f-vector [18–21].

For simplicial $(d-1)$-complex Δ, define the polynomials

$$f(\Delta,t)=\sum_{j=-1}^{d-1} f_j(\Delta)t^{j+1}$$

and

$$h(\Delta, t) = (1-t)^d f\left(\Delta, \frac{t}{1-t}\right).$$

The $(d+1)$-vector $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$ is determined by the polynomial relation

$$h(\Delta, t) = \sum_{i=0}^{d} h_i(\Delta) t^i$$

and is called the *h-vector* of Δ. Set $h_i(\Delta) = 0$ if $i < 0$ or $i > d$. (McMullen and Walkup [16] write $g_{i-1}^{(d)}(\Delta)$ instead of $h_i(\Delta)$, and use $f(\Delta, t) = \sum_{j=-1}^{d-1} (-1)^{j+1} f_j(\Delta) t^{j+1}$ and $g^{(d)}(\Delta, t) = (1-t)^d f(\Delta, t/t-1) = \sum_{i=-1}^{d-1} g_i^{(d)}(\Delta) t^{i+1}$ to define the $g_i^{(d)}(\Delta)$.)

From the definition of $h(\Delta)$, one can explicitly write the $h_i(\Delta)$ as linear combinations of the $f_j(\Delta)$:

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\Delta), \qquad 0 \leq i \leq d.$$

Moreover, $f(\Delta)$ can be recovered from $h(\Delta)$ by

$$f(\Delta, t) = (1+t)^d h\left(\Delta, \frac{t}{1+t}\right),$$

from which the $f_j(\Delta)$ can be explicitly expressed as non-negative linear combinations of the $h_i(\Delta)$:

$$f_j(\Delta) = \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_i(\Delta), \qquad -1 \leq j \leq d-1.$$

Thus we have a bijection between the f-vectors and the h-vectors of simplicial $(d-1)$-complexes. For a simplicial complex Δ it is also useful to define $g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta)$ for all integer i.

3. Polyhedral Complexes

For each simple d-polyhedron P we wish to describe a simplicial $(d-1)$-complex $\Sigma^*(P)$ that is *dual* to P in the sense that there is an inclusion-reversing bijection between the set of faces of $\Sigma^*(P)$ and the set of non-empty faces of P, matching j-faces of $\Sigma^*(P)$ with $(d-j-1)$-faces of P, $-1 \leq j \leq d-1$. Such a simplicial complex will be called a *(simplicial) polyhedral $(d-1)$-complex.*

First, suppose P is a simple d-polytope. Let P^* be a simplicial d-polytope dual to P and define

$$\Sigma(P^*) = \{F \subset V(P^*) \colon \text{conv } F \text{ is a face of } \partial P^*\},$$

where conv F denotes the convex hull of F. Then the desired simplicial complex $\Sigma^*(P)$ is precisely $\Sigma(P^*)$. In this case

$$f_j(P) = f_{d-j-1}(\Sigma(P^*)), \qquad 0 \leq j \leq d-1.$$

Because $\Sigma(P^*)$ is a simplicial $(d-1)$-sphere, we will say that it is a *(simplicial) polyhedral $(d-1)$-sphere.*

Now suppose P is an unbounded simple d-polyhedron. Then there exists a simple d-polytope Q with facet F such that P is combinatorially equivalent to $Q \sim F$, the unbounded simple d-polyhedron obtained from Q by applying a projective transformation that sends F into the hyperplane at infinity [7, §1.1; 11; 15, § 1.2]. Let Q^* be a simplicial d-polytope dual to Q and v be the vertex of Q^* corresponding to F under duality. Then the desired simplicial complex $\Sigma^*(P)$ will be $\Sigma(Q^*)\backslash v$. In fact, $\Sigma(Q^*)\backslash v$ is a simplicial $(d-1)$-ball, so we will refer to it as a *(simplicial) polyhedral* $(d-1)$*-ball*. In this case,

$$f_j(Q)=f_{d-j-1}(\Sigma(Q^*)), \qquad 0\leqslant j\leqslant d-1,$$
$$f_j(P)=f_{d-j-1}(\Sigma(Q^*)\backslash v), \qquad 0\leqslant j\leqslant d-1,$$
$$f_j(F)=f_{j+1}^{(u)}(P)=f_{d-j-2}(\mathrm{lk}_{\Sigma(Q^*)}v), \qquad 0\leqslant j\leqslant d-2,$$

where F itself is regarded as a simple $(d-1)$-polytope and $f_j^{(u)}(P)$ is the number of unbounded j-faces of P.

We proceed to develop some properties of polyhedral complexes, and will sketch or omit the proofs of the more straightforward results. In the case that Δ is a polyhedral $(d-1)$-sphere, say, $\Delta=\Sigma(P)$ for some simplicial d-polytope P, we will set $f_{-1}(P)=1$ and $f_j(P)=0$ if $j<-1$ or $j>d-1$, and write $f(P,t)$, $h(P,t)$, $h_i(P)$, $h(P)$ and $g_i(P)$ for $f(\Delta,t)$, $h(\Delta,t)$, $h_i(\Delta)$, $h(\Delta)$ and $g_i(\Delta)$, respectively. We call $h(P)$ the *h-vector* of P and define

$$h(\mathscr{P}_s^d)=\{h(P)\colon P\in\mathscr{P}_s^d\}.$$

The first three lemmas concern some basic relationships among polyhedral complexes. Suppose P is a simplicial d-polytope, $d\geqslant 1$, and $v\in V(P)$. Let H be a hyperplane strictly separating v from the remaining vertices of P; i.e. $\{v\}$ and $V(P)\backslash v$ are contained within opposite open half-spaces determined by H. Then the simplicial $(d-1)$-polytope $Q=P\cap H$ is a *vertex figure* of P at v. In fact $\Sigma(Q)\cong\mathrm{lk}_{\Sigma(P)}v$, so we have the following.

LEMMA 3.1. *If $\Sigma(P)$ is a polyhedral $(d-1)$-sphere and v is a vertex of $\Sigma(P)$, then $\mathrm{lk}_{\Sigma(P)}v$ is a polyhedral $(d-2)$-sphere.*

Moreover, $\mathrm{lk}_{\Sigma(P)}v$ is closely related to the polyhedral ball $\Sigma(P)\backslash v$.

LEMMA 3.2. *Let $\Sigma(P)$ and v be as above. Then $\partial(\Sigma(P)\backslash v)=\mathrm{lk}_{\Sigma(P)}v$.*

COROLLARY 3.3. *The boundary of a polyhedral $(d-1)$-ball is a polyhedral $(d-2)$-sphere.*

We now turn to some specific examples of polyhedral complexes and mention a few operations on simplicial complexes that preserve the property of being polyhedral. To begin, let $\Delta=\bar{F}$ for some finite set F of cardinality $k\geqslant 1$. Then Δ is a simplicial $(k-1)$-complex and is called a *(combinatorial)* $(k-1)$*-simplex*.

LEMMA 3.4. *Let $k\geqslant 1$ and Δ be a $(k-1)$-simplex. Then Δ is a polyhedral $(k-1)$-ball, $\partial\Delta=\Delta\backslash F$ is a polyhedral $(k-2)$-sphere* and

(i) $h_i(\Delta)=\begin{cases}1, & i=0\\ 0, & 1\leqslant i\leqslant k.\end{cases}$

(ii) $h_i(\partial\Delta)=1, \quad 0\leqslant i\leqslant k-1.$

PROOF. Use $f(\Delta,t)=(1+t)^k$ and $f(\partial\Delta,t)=(1+t)^k-t^k$.

Let Δ_1 and Δ_2 be simplicial complexes of dimension d_1-1 and d_2-1 on disjoint vertex sets $V(\Delta_1)$ and $V(\Delta_2)$, respectively. The *join* of Δ_1 and Δ_2, denoted $\Delta_1\cdot\Delta_2$, is the simplicial

(d_1+d_2-1)-complex on the set $V(\Delta_1)\cup V(\Delta_2)$ defined by

$$\Delta_1\cdot\Delta_2=\{F_1\cup F_2: F_1\in\Delta_1, F_2\in\Delta_2\}.$$

LEMMA 3.5. *If Δ_1 and Δ_2 are as above, then*

$$h(\Delta_1\cdot\Delta_2, t)=h(\Delta_1, t)h(\Delta_2, t).$$

Further, if Δ_1 and Δ_2 are polyhedral complexes, then so is $\Delta_1\cdot\Delta_2$. In particular, if Δ is a polyhedral $(d-1)$-complex and $v\notin V(\Delta)$, then $\bar{v}\cdot\Delta$ is a polyhedral d-ball and

$$h_i(\bar{v}\cdot\Delta)=\begin{cases}h_i(\Delta), & 0\leq i\leq d\\ 0, & i=d+1.\end{cases}$$

PROOF. Use $f(\Delta_1\cdot\Delta_2, t)=f(\Delta_1, t)f(\Delta_2, t)$ and the fact that the joining of two polyhedral complexes is dual to the operation of taking the Cartesian product of two simple polyhedra [12, 17].

Suppose Δ is a polyhedral $(d-1)$-sphere, $d\geq 1$, and F is a facet of Δ. For $v\notin V(\Delta)$, the *stellar subdivision* of F in Δ is the simplicial complex

$$\mathrm{st}(v, F)[\Delta]=(\Delta\setminus F)\cup\bar{v}\cdot\partial\bar{F}.$$

LEMMA 3.6. *If Δ is as above, then* $\mathrm{st}(v, F)[\Delta]$ *is a polyhedral $(d-1)$-sphere and*

$$h_i(\mathrm{st}(v, F)[\Delta])=\begin{cases}h_i(\Delta), & i=0 \text{ or } d\\ h_i(\Delta)+1, & 1\leq i\leq d-1.\end{cases}$$

PROOF. If $\Delta=\Sigma(P)$ for simplicial d-polytope P, then $\mathrm{st}(v, F)[\Delta]=\Sigma(Q)$, where Q is obtained from P by building a pyramidal cap on the facet of P corresponding to F[7, p. 217; 17].

Now let Δ be a polyhedral $(d-1)$-sphere, $d\geq 1$, and $v\in V(\Delta)$. For $u\notin V(\Delta)$, the *co-wedge* of Δ on v is the simplicial complex

$$w(u, v)[\Delta]=\{\varnothing, u, v\}\cdot(\Delta\setminus v)\cup\overline{uv}\cdot\mathrm{lk}_\Delta v.$$

LEMMA 3.7. *If Δ is as above, then $w(u, v)[\Delta]$ is a polyhedral d-sphere and*
(i) $h_i(w(u, v)[\Delta])=h_{i-1}(\Delta)+h_i(\Delta)-h_{i-1}(\mathrm{lk}_\Delta v)$, $0\leq i\leq d+1$.
(ii) $\mathrm{lk}_{w(u,v)[\Delta]}u=\Delta$.

PROOF. The operation of co-wedge is dual to the operation of the wedge of a simple polytope on a facet [12, 17].

We now turn to some further properties of the h-vectors of polyhedral complexes. One advantage of the h-vector over the f-vector is that it concisely represents the set of linear relations satisfied by the f_j of simplicial $(d-1)$-spheres known as the *Dehn–Sommerville equations.*

THEOREM 3.8 (DEHN–SOMMERVILLE EQUATIONS). *If Δ is a simplicial $(d-1)$-sphere, then*

$$\sum_{j=k}^{d-1}(-1)^j\binom{j+1}{k+1}f_j(\Delta)=(-1)^{d-1}f_k(\Delta),\ -1\leq k\leq d-2.$$

Equivalently,

$$h_i(\Delta)=h_{d-i}(\Delta), \qquad 0\leq i\leq [d/2],$$

where $[d/2]$ *is the greatest integer not exceeding* $d/2$.

PROOF. See [7, Ch. 9; 8; 15, § 5.1; 16].

The Dehn–Sommerville equations allow us to derive some interesting relationships among P, $\Delta=\Sigma(P)\backslash v$, and $\partial\Delta=\mathrm{lk}_{\Sigma(P)}v$, for simplicial d-polytope P, $d\geq 1$, and $v\in V(P)$. Specifically, $h(P)$ and $h(\partial\Delta)$ are each completely determined by $h(\Delta)$:

COROLLARY 3.9 (MCMULLEN–WALKUP [16]). *Let* P *and* Δ *be as above. Then*
(i) $f_j(P)=f_j(\Delta)+f_{j-1}(\partial\Delta)$, $-1\leq j\leq d-1$.
(ii) $h_i(P)=h_i(\Delta)+h_{i-1}(\partial\Delta)$, $0\leq i\leq d$.
(iii) $g_i(\partial\Delta)=h_i(\Delta)-h_{d-i}(\Delta)$, $0\leq i\leq d$.
(iv) $g_i(P)=h_i(\Delta)-h_{d-i+1}(\Delta)$, $0\leq i\leq d+1$.
(v) $f_k(\partial\Delta)=f_k(\Delta)+(-1)^d\sum_{j=k}^{d-1}(-1)^j\binom{j+1}{k+1}f_j(\Delta)$, $-1\leq k\leq d-2$.

PROOF. Part (i) is clear from the fact that $\Sigma(P)=\Delta\cup\bar{v}\cdot\partial\Delta$, from which we deduce $f(P,t)=f(\Delta,t)+tf(\partial\Delta,t)$, which in turn leads to (ii). For (iii) and (iv), use (ii) and apply the Dehn–Sommerville equations to P and $\partial\Delta$:

$$\begin{aligned}h_i(\Delta)-h_{d-i}(\Delta)&=h_i(P)-h_{d-i}(P)-h_{i-1}(\partial\Delta)+h_{d-i-1}(\partial\Delta)\\&=h_i(\partial\Delta)-h_{i-1}(\partial\Delta)\\&=g_i(\partial\Delta), \qquad 0\leq i\leq d,\end{aligned}$$

and

$$\begin{aligned}h_i(\Delta)-h_{d-i+1}(\Delta)&=h_i(P)-h_{d-i+1}(P)-h_{i-1}(\partial\Delta)+h_{d-i}(\partial\Delta)\\&=h_i(P)-h_{i-1}(P)\\&=g_i(P), \qquad 0\leq i\leq d+1.\end{aligned}$$

Now (iii) is equivalent to

$$(1-t)h(\partial\Delta,t)=h(\Delta,t)-t^dh(\Delta,t^{-1}),$$

from which we obtain

$$f(\partial\Delta,t)=f(\Delta,t)+(-1)^{d-1}f(\Delta,-1-t)$$

and equating coefficients of t^{k+1} yields (v).

COROLLARY 3.10. *Let* P *be a simple* d*-polyhedron. Then the number of unbounded* k*-faces of* P *is given by*

$$f_k^{(u)}(P)=f_k(P)-\sum_{j=0}^{k}(-1)^j\binom{d-j}{d-k}f_j(P), \qquad 1\leq k\leq d-1.$$

PROOF. If P is bounded, $f_k^{(u)}(P)=0$ and the above statement follows from the Dehn–Sommerville equations for P^*. If P is unbounded, let $\Delta=\Sigma^*(P)$ and use $f_j(P)=f_{d-j-1}(\Delta)$ and $f_j^{(u)}(P)=f_{d-j-1}(\partial\Delta)$, $1\leq k\leq d$. Then the result is a consequence of Corollary 3.9(v).

COROLLARY 3.11. *Let $P \subseteq \mathbf{R}^d$ be a simplicial d-polytope and $z \in \mathbf{R}^d$ be a point beyond a non-empty set $\mathscr{B}$ of facets of P and beneath the remainder of the facets of P. Define the simplicial complex Δ to be*

$$\Delta = \cup\{\bar{F} : F \in \Sigma(P), \operatorname{conv} F \in \mathscr{B}\}.$$

Let Q be the simplicial d-polytope $Q = \operatorname{conv}(P \cup \{z\})$. Then

$$h_i(Q) = h_i(P) + h_{i-1}(\partial\Delta) - h_{d-i}(\Delta), \qquad 0 \leq i \leq d.$$

Note. For the notions of *beneath* and *beyond*, see [7, § 5.2]. As a consequence of [6], Δ must be a simplicial $(d-1)$-ball.

PROOF. First observe [7, § 5.2] that

$$\begin{aligned} f_j(Q) &= f_j(P) - f_j(\Delta^\circ) + f_{j-1}(\partial\Delta) \\ &= f_j(P) - f_j(\Delta) + f_j(\partial\Delta) + f_{j-1}(\partial\Delta), \qquad -1 \leq j \leq d-1. \end{aligned}$$

From this we obtain

$$\begin{aligned} f(Q, t) &= f(P, t) - f(\Delta, t) + (1+t) f(\partial\Delta, t), \\ h(Q, t) &= h(P, t) - h(\Delta, t) + h(\partial\Delta, t), \\ h_i(Q) &= h_i(P) - h_i(\Delta) + h_i(\partial\Delta), \ 0 \leq i \leq d. \end{aligned}$$

The corollary now follows from 3.9(iii), since

$$\begin{aligned} h_i(\Delta) &= g_i(\partial\Delta) + h_{d-i}(\Delta) \\ &= h_i(\partial\Delta) - h_{i-1}(\partial\Delta) + h_{d-i}(\Delta), \ 0 \leq i \leq d. \end{aligned}$$

The remainder of this section will be devoted to a brief discussion of the characterization of $h(\mathscr{P}_s^d)$ and some related constructions and corollaries. For positive integers k and i, k can be written uniquely in the form

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$, from which is defined

$$k^{<i>} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}.$$

We define also $0^{\langle i \rangle} = 0$ for every positive integer i. A $(d+1)$-vector of integers $(h_0, h_1, \ldots, h_d)$ is called an *O-sequence* (or *M-vector*) if $h_0 = 1$, $h_i \geq 0$, $1 \leq i \leq d$, and $h_{i+1} \leq h_i^{\langle i \rangle}$, $1 \leq i \leq d-1$ [19, 20].

THEOREM 3.12 (McMULLEN'S CONDITIONS). *Let $h = (h_0, h_1, \ldots, h_d)$ be a $(d+1)$-vector of integers, $g_0 = h_0$, and $g_i = h_i - h_{i-1}$, $1 \leq i \leq n = [d/2]$. Then $h \in h(\mathscr{P}_s^d)$ if and only if the following two conditions hold*:

(i) $h_i = h_{d-i}$, $0 \leq i \leq n$.

(ii) $(g_0, g_1, \ldots, g_n)$ *is an O-sequence.*

PROOF. See [3, 4, 13, 21].

By examining the method of construction presented in [3, 4, 13], we can actually make a stronger statement about the existence of a simplicial polytope with a specified h-vector.

THEOREM 3.13. *Let $d \geq 1$ and $h = (h_0, h_1, \ldots, h_d) \in h(\mathcal{P}_s^d)$. Choose any integer $\nu \geq h_1 + d$. Then there exist a simplicial d-polytope P and a simplicial $(d+1)$-polytope Q such that:*

(i) *Q has $\nu + 1$ vertices and P is a vertex figure of Q at some vertex $z \in V(Q)$;*

(ii) *$h(P) = h$;*

(iii) *there is a vertex v of P such that $h_i(\mathrm{lk}_{\Sigma(P)} v) = h_i$, $0 \leq i \leq [(d-1)/2]$;*

(iv) *$h_i(Q) = \binom{\nu - d + i - 2}{i} + h_{i-1}$, $0 \leq i \leq [(d+1)/2]$.*

PROOF. Define $g_0 = h_0$ and $g_i = h_i - h_{i-1}$, $0 \leq i \leq n = [d/2]$. The case $d = 1$ is easily checked, so assume $d \geq 2$. If $\nu = d+1$, then $h_1 = 1$ and $g_1 = 0$. Condition (ii) of 3.12 then forces $g_i = 0$, $1 \leq i \leq n$. Thus $h_i = 1$, $0 \leq i \leq d$. Let Q be any geometric $(d+1)$-simplex, z be any vertex of Q, P be a vertex figure of Q at z, and v be any vertex of P. Then $\Sigma(Q)$ is the boundary of a $(d+1)$simplex, $\Sigma(P)$ is the boundary of a d-simplex, and $\mathrm{lk}_{\Sigma(P)} v$ is the boundary of a $(d-1)$-simplex. Therefore (i) through (iv) hold, using 3.4(ii).

Now suppose $d \geq 2$ and $\nu > d+1$. As remarked in [4, 13], slight modifications in the construction yield (i) and (ii), whether $\nu = h_1 + d$ or $\nu > h_1 + d$. In fact, we take $Q = \mathrm{conv}(C(\nu, d+1) \cup \{z\})$, where $C(\nu, d+1)$ is the cyclic polytope in $\mathbf{R}^{d+1}$ with ν vertices and $z \in \mathbf{R}^{d+1}$ is beyond a certain set $\mathcal{B}$ of facets of $C(\nu, d+1)$ and beneath the rest. Recall from [4, 13] that the simplicial d-complex Δ associated with $\mathcal{B}$ is a simplicial d-ball, $\partial\Delta \cong \Sigma(P)$, $h(\partial\Delta) = h$, and

$$h_i(\Delta) = \begin{cases} g_i, & 0 \leq i \leq n \\ 0, & n+1 \leq i \leq d+1. \end{cases}$$

Now in fact

$$h_i(C(\nu, d+1)) = \binom{\nu - d + i - 2}{i}, \quad 0 \leq i \leq [(d+1)/2],$$

[15] and so (iv) follows immediately from 3.11.

In order to demonstrate (iii), we note that Δ is of the form $\Delta = \bar{v}_1 \cdot \Delta_1$, for some simplicial $(d-1)$-complex Δ_1, and hence $\Sigma(P) \backslash v_1 \cong (\partial\Delta) \backslash v_1 = \Delta_1$. Therefore Δ_1 is a polyhedral $(d-1)$-ball, and it is a simple matter to verify that $\mathrm{lk}_{\partial\Delta} v_1 = \partial \mathrm{lk}_{\Delta} v_1 = \partial\Delta_1$. By 3.5,

$$h_i(\Delta) = \begin{cases} h_i(\Delta_1), & 0 \leq i \leq d \\ 0, & i = d+1 \end{cases}$$

and thus

$$h_i(\Delta_1) = \begin{cases} g_i, & 0 \leq i \leq n \\ 0, & n+1 \leq i \leq d. \end{cases}$$

Now 3.9(iii) forces $g_i(\partial\Delta_1) = g_i$, $0 \leq i \leq [(d-1)/2]$, from which we conclude $h_i(\partial\Delta_1) = h_i$, $0 \leq i \leq [(d-1)/2]$. Finally, we compute

$$\begin{aligned} h_i(\mathrm{lk}_{\Sigma(P)} v_1) &= h_i(\mathrm{lk}_{\partial\Delta} v_1) \\ &= h_i(\partial \mathrm{lk}_{\Delta} v_1) \\ &= h_i(\partial\Delta_1) \\ &= h_i, \quad 0 \leq i \leq [(d-1)/2], \end{aligned}$$

verifying (iii).

Although as yet we do not have a characterization of $\{h(\Delta) \colon \Delta$ is a polyhedral $(d-1)$-ball$\}$, Theorem 3.12 does lead to a strong necessary condition for such h-vectors:

COROLLARY 3.14. *Let Δ be a polyhedral $(d-1)$-ball. Then (recalling the convention that $h_i(\Delta)=0$ if $i>d$),*

$$(h_0(\Delta)-h_{d+k}(\Delta), h_1(\Delta)-h_{d+k-1}(\Delta), \ldots, h_m(\Delta)-h_{d+k-m}(\Delta))$$

is an O-sequence for all integer $0\leq k\leq d+1$, where $m=[(d+k-1)/2]$. In particular, $h_i(\Delta)\leq\binom{\nu-d+i-1}{i}$, $1\leq i\leq d$, where $\nu=f_0(\Delta)$; $h_i(\Delta)\geq h_{d-i}(\Delta)$, $0\leq i\leq[d/2]$; and $h_i(\Delta)\geq h_{i+1}(\Delta)$, $[d/2]\leq i\leq d-1$.

PROOF. Set $\Delta_0=\Delta$ and for $k\geq 1$ put $\Delta_k=\bar{u}_k\cdot\Delta_{k-1}$ for some $u_k\notin V(\Delta_{k-1})$. By repeated application of 3.5, Δ_k is a polyhedral $(d+k-1)$-ball and $h_i(\Delta_k)=h_i(\Delta)$, $0\leq i\leq d+k$. By 3.3, $\partial\Delta_k$ is a polyhedral $(d+k-2)$-sphere, and 3.9(iii) implies $g_i(\partial\Delta_k)=h_i(\Delta_k)-h_{d+k-i}(\Delta_k)=h_i(\Delta)-h_{d+k-i}(\Delta)$, $0\leq i\leq d+k$. Now 3.12(ii) implies $(g_0(\partial\Delta_k), g_1(\partial\Delta_k), \ldots, g_m(\partial\Delta_k))$ is an O-sequence.

In particular, taking $k=d+1$, we see that $(h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$ is an O-sequence, where $h_1=\nu-d$. Then it is immediate [18] that

$$h_i(\Delta)\leq\binom{\nu-d+i-1}{i},\ 1\leq i\leq d.$$

On the other hand, taking $k=0$ implies in particular that $h_i(\Delta)-h_{d-i}(\Delta)\geq 0$, i.e. $h_i(\Delta)\geq h_{d-i}(\Delta)$, $0\leq i\leq[d/2]$. Finally, if we select i such that $[d/2]\leq i\leq d-1$ and examine $k=2i-d+1$, then $d+k-i=i+1$ and $[(d+k-1)/2]=i$ so $h_i(\Delta_k)\geq 0$ means $h_i(\Delta)\geq h_{i+1}(\Delta)$.

4. POLYTOPE PAIRS

Klee [9] in 1966 proved that every simple d-polyhedron with ν facets has at least $\nu-d+1$ vertices. Grünbaum [7, § 10.2] speculated whether this result might be improved upon if one specified both the number of bounded and of unbounded facets. In 1974, Klee [10, 11] approached problems of this form from the point of view of pairs of simple polytopes. In this section we look at polytopes in a dual fashion to that of Klee, and strengthen and extend some of his results.

We first note that Klee's result above can now easily be proved in the dual context of polyhedral complexes.

PROPOSITION 4.1. *If Δ is a polyhedral $(d-1)$-complex with ν vertices, then $f_{d-1}(\Delta)\geq\nu-d+1$.*

PROOF. First, $h(\Delta)$ satisfies $h_0(\Delta)=1$ and $h_1(\Delta)=\nu-d$. If Δ is a polyhedral $(d-1)$-sphere, then $h_i(\Delta)\geq 0$, $2\leq i\leq d$, by 3.12. If Δ is a polyhedral $(d-1)$-ball, then $h_i(\Delta)\geq 0$, $2\leq i\leq d$, by 3.14. In either case, $f_{d-1}(\Delta)=\sum_{i=0}^{d}h_i(\Delta)\geq\nu-d+1$.

A *polytope pair* (P, v) of type (d, ν, h), where $1\leq d<\nu$ and $h\in h(\mathscr{P}_s^{d-1})$, is a simplicial polytope P and a vertex $v\in V(P)$ such that $f_0(P)=\nu$ and $h=h(\mathrm{lk}_{\Sigma(P)}v)$. Note that if (P, v) is a polytope pair of type (d, ν, h), then $1\leq h_1=f_0(\mathrm{lk}_{\Sigma(P)}v)-d+1\leq\nu-d$. Define

$$\lambda_i^1(d, \nu, h)=\min h_i(P),\qquad 0\leq i\leq d,$$

$$\lambda_i^2(d, \nu, h)=\min h_i(\Sigma(P)\backslash v),\qquad 0\leq i\leq d,$$

$$\mu_i^1(d, \nu, h)=\max h_i(P),\qquad 0\leq i\leq d,$$

$$\mu_i^2(d, \nu, h)=\max h_i(\Sigma(P)\backslash v),\qquad 0\leq i\leq d,$$

as (P, v) ranges over the set of all polytope pairs of type (d, ν, h). Of course, the Dehn–Sommerville equations imply that

$$\lambda_i^1(d, \nu, h) = \lambda_{d-i}^1(d, \nu, h), \qquad 0 \leq i \leq [d/2],$$
$$\mu_i^1(d, \nu, h) = \mu_{d-i}^1(d, \nu, h), \qquad 0 \leq i \leq [d/2].$$

The main result of this paper is the following.

THEOREM 4.2. *Let $3 \leq d < \nu$ and $h \in h(\mathscr{P}_s^{d-1})$ such that $1 \leq h_1 \leq \nu - d$. Put $n = [d/2]$ and $m = [(d-1)/2]$. Then*

(i) $$\lambda_i^1(d, \nu, h) = \begin{cases} 1, & i = 0 \\ \nu - d - h_1 + h_i, & 1 \leq i \leq n. \end{cases}$$

(ii) $$\lambda_i^2(d, \nu, h) = \begin{cases} 1, & i = 0 \\ \nu - d - h_1 + h_i - h_{i-1}, & 1 \leq i \leq m \\ \nu - d - h_1, & m+1 \leq i \leq d-1 \\ 0, & i = d. \end{cases}$$

(iii) $$\mu_i^1(\mathrm{d}, \nu, h) = \binom{\nu - d + i - 2}{i} + h_{i-1}, \qquad 0 \leq i \leq n.$$

(iv) $$\mu_i^2(d, \nu, h) = \begin{cases} \binom{\nu - d + i - 2}{i}, & 0 \leq i \leq n \\ \binom{\nu - i - 2}{d - i} + h_i - h_{i-1}, & n+1 \leq i \leq d. \end{cases}$$

Moreover, there exist polytope pairs (P_1, v_1) and (P_2, v_2) of type (d, ν, h) such that

$$h_i(P_1) = \lambda_i^1(d, \nu, h), \qquad 0 \leq i \leq d,$$
$$h_i(\Sigma(P_1)\backslash v_1) = \lambda_i^2(d, \nu, h), \qquad 0 \leq i \leq d,$$
$$h_i(P_2) = \mu_i^1(d, \nu, h), \qquad 0 \leq i \leq d,$$
$$h_i(\Sigma(P_2)\backslash v_2) = \mu_i^2(d, \nu, h), \qquad 0 \leq i \leq d.$$

PROOF. Note that $d = n + m + 1$ and recall by the Dehn–Sommerville equations that $h_i(P) = h_{d-i}(P)$, $0 \leq i \leq d$, and $h_i = h_{d-i-1}$, $0 \leq i \leq d-1$, for every polytope pair of type (d, ν, h).

Establishing the bounds. Let (P, v) be a polytope pair of type (d, ν, h) and put $\Delta = \Sigma(P)\backslash v$. Recall (Lemma 3.2) that $\partial\Delta = \mathrm{lk}_{\Sigma(P)} v$. Then $h_0(\Delta) = 1$ and $h_1(\Delta) = \nu - d - 1$. From 3.9(iii), $h_i(\Delta) - h_{d-i}(\Delta) = h_i - h_{i-1}$; in particular $h_d(\Delta) = h_0(\Delta) - h_0 + h_{-1} = 0$ and $h_{d-1}(\Delta) = h_1(\Delta) - h_1 + h_0 = \nu - d - h_1$. Then 3.14 implies $h_i(\Delta) \geq \nu - d - h_1$, $m+1 \leq i \leq d-2$. Therefore

$$\lambda_i^2(d, \nu, h) \geq \nu - d - h_1, \qquad m+1 \leq i \leq d-1.$$

For $1 \leq i \leq m$, using 3.9(iii) we have

$$\begin{aligned} h_i(\Delta) &= h_{d-i}(\Delta) + h_i - h_{i-1} \\ &\geq \nu - d - h_1 + h_i - h_{i-1}. \end{aligned}$$

Therefore

$$\lambda_i^2(d, \nu, h) \geq \nu - d - h_1 + h_i - h_{i-1}, \qquad 1 \leq i \leq m.$$

Using 3.9(ii), for $1 \leqslant i \leqslant n$ we have

$$\begin{aligned} h_i(P) &= h_{d-i}(P) \\ &= h_{d-i}(\Delta) + h_{d-i-1} \\ &\geqslant \nu - d - h_1 + h_i. \end{aligned}$$

Therefore

$$\lambda_i^1(d, \nu, h) \geqslant \nu - d - h_1 + h_i, \qquad 1 \leqslant i \leqslant n.$$

To place bounds on the μ_i, recall first by 3.14 that

$$h_i(\Delta) \leqslant \binom{\nu - d + i - 2}{i}, \; 2 \leqslant i \leqslant n.$$

Therefore

$$\mu_i^2(d, \nu, h) \leqslant \binom{\nu - d + i - 2}{i}, \qquad 2 \leqslant i \leqslant n.$$

Choose $n+1 \leqslant i \leqslant d$. Using 3.9(iii) we obtain

$$\begin{aligned} h_i(\Delta) &= h_{d-i}(\Delta) + h_i - h_{i-1} \\ &\leqslant \binom{\nu - i - 2}{d - i} + h_i - h_{i-1}. \end{aligned}$$

Therefore

$$\mu_i^2(d, \nu, h) \leqslant \binom{\nu - i - 2}{d - i} + h_i - h_{i-1}, \qquad n+1 \leqslant i \leqslant d.$$

Again by 3.9(ii)

$$h_i(P) = h_i(\Delta) + h_{i-1}, \qquad 0 \leqslant i \leqslant n, \quad \text{so} \quad h_i(P) \leqslant \binom{\nu - d + i - 2}{i} + h_{i-1}, \qquad 0 \leqslant i \leqslant n.$$

Therefore

$$\mu_i^1(d, \nu, h) \leqslant \binom{\nu - d + i - 2}{i} + h_{i-1}, \; 0 \leqslant i \leqslant n.$$

Achieving the bounds. Suppose we have found polytope pairs (P_1, v_1) and (P_2, v_2) of type (d, ν, h) such that $h(P_1)$ achieves all of the lower bounds for $\lambda_i^1(d, \nu, h)$ simultaneously and $h(P_2)$ achieves all of the upper bounds for $\mu_i^1(d, \nu, h)$ simultaneously. Because the bounds for $h_i(P)$ were obtained from those for $h_i(\Delta)$ using 3.9(ii), we would then have $h(\Sigma(P_1)\backslash v_1)$ and $h(\Sigma(P_2)\backslash v_2)$ achieving all of the bounds for $\lambda_i^2(d, \nu, h)$ and $\mu_i^2(d, \nu, h)$, respectively.

By 3.13 there exists a simplicial $(d-1)$-polytope P such that

(i) $h(P) = h$;

(ii) there is a vertex v of P such that $h_i(\mathrm{lk}_{\Sigma(P)} v) = h_i$, $0 \leqslant i \leqslant [(d-2)/2] = n-1$. Let $\Sigma = w(u, v)[\Sigma(P)]$ for some $u \notin V(P)$. By 3.7, for $0 \leqslant i \leqslant n$,

$$\begin{aligned} h_i(\Sigma) &= h_{i-1}(P) + h_i(P) - h_{i-1}(\mathrm{lk}_{\Sigma(P)} v) \\ &= h_{i-1} + h_i - h_{i-1} \\ &= h_i. \end{aligned}$$

Now 3.7 also implies Σ is a polyhedral $(d-1)$-sphere with h_1+d vertices. Note that $\mathrm{lk}_\Sigma u=\Sigma(P)$. Put $\Sigma_0=\Sigma$ and for integer $j\geqslant 1$ let $\Sigma_j=\mathrm{st}(u_j, F_{j-1})[\Sigma_{j-1}]$ for some $u_j\notin V(\Sigma_{j-1})$ and some facet F_{j-1} of Σ_{j-1} not containing u. Then Σ_j is a polyhedral $(d-1)$-sphere by 3.6 and

$$\begin{aligned}\mathrm{lk}_{\Sigma_j}u&=\mathrm{lk}_{(\Sigma_{j-1}\setminus F_{j-1})\cup \bar{u}_j\cdot\partial\bar{F}_{j-1}}u\\&=\mathrm{lk}_{(\Sigma_{j-1}\setminus F_{j-1})}u\\&=\mathrm{lk}_{\Sigma_{j-1}}u\\&=\Sigma(P),\end{aligned}$$

applying induction on $j\geqslant 0$. Let P_1 be a simplicial d-polytope such that $\Sigma(P_1)\cong\Sigma_{\nu-d-h_1}$, and put $v_1=u$. By repeated use of 3.6, $h_1(\Sigma(P_1))=h_1+(\nu-d-h_1)=\nu-d$, and so $f_0(P_1)=\nu$. Thus (P_1, v_1) is a polytope pair of type (d, ν, h). Also by 3.6, for $1\leqslant i\leqslant n$,

$$\begin{aligned}h_i(P_1)&=h_i(\Sigma_0)+\nu-d-h_1\\&=h_i+\nu-d-h_1.\end{aligned}$$

Therefore $h(P_1)$ achieves all of the lower bounds for $\lambda_i^1(d, \nu, h)$ simultaneously, and so $h(\Sigma(P_1)\setminus v_1)$ achieves all of the lower bounds for $\lambda_i^2(d, \nu, h)$ simultaneously.

For (iii) and (iv), by 3.13 there exist $P\in\mathscr{P}_s^{d-1}$ and $Q\in\mathscr{P}_s^d$ such that

(i) $h(P)=h$;

(ii) Q has ν vertices and P is a vertex figure of Q at a vertex $z\in V(Q)$;

(iii) $h_i(Q)=\binom{\nu-d+i-2}{i}+h_{i-1},\ 0\leqslant i\leqslant n.$

Put $P_2=Q$ and $v_2=z$. Then (P_2, v_2) is a polytope pair of type (d, ν, h) such that $h(P_2)$ achieves all of the upper bounds for $\mu_i^1(d, \nu, h)$ simultaneously, and so $h(\Sigma(P_2)\setminus v_2)$ achieves all of the upper bounds for $\mu_i^2(d, \nu, h)$ simultaneously.

We remark that the maxima of Theorem 4.2 still hold if the notion of a polytope pair (P, v) of type (d, ν, h) is extended to that of *pointed sphere* (S, v) of type (d, ν, h), where S is a simplicial $(d-1)$-sphere with ν vertices and v is a vertex of S with $h=h(\mathrm{lk}_S v)$. To see this, note first that Corollary 3.9 remains true in the case where $P=S$, $\Delta=S\setminus v$, and $\partial\Delta=\mathrm{lk}_S v$, since the Dehn–Sommerville equations continue to hold for S and $\mathrm{lk}_S v$ by [8] (the latter being an homology $(d-2)$-sphere). Because $\Delta=S\setminus v$ is an homology $(d-1)$-ball, we know by [18] that

$$h_i(\Delta)\leqslant\binom{\nu-d+i-2}{i},\ 2\leqslant i\leqslant n,$$

and hence we establish the same upper bounds as in Theorem 4.2. At the present time, there appears to be no straightforward way to extend the lower bounds of this theorem to pointed spheres as well.

Suppose $3\leqslant d\leqslant r<\nu$ are integers. Let us abuse notation slightly and say that a polytope pair (P, v) is of type (d, ν, r) if it is of type (d, ν, h) for some $h\in h(\mathscr{P}_s^{d-1})$ such that $h_1=r-d+1$; i.e. if P is a simplicial d-polytope with ν vertices, one of which, v, is on exactly r edges.

COROLLARY 4.3. *Let $3\leqslant d\leqslant r<\nu$ and put $n=[d/2]$ and $m=[(d-1)/2]$. Then as (P, v) ranges over all polytope pairs of type (d, ν, r) we have the following minima and* *maxima*:

$$
\begin{array}{lll}
 & \textit{function} & \textit{minimum} \\
\text{(i)} & h_i(\mathrm{lk}_{\Sigma(P)}v) & \begin{cases} 1, & i=0, \\ r-d+1. & 1\le i\le m, \end{cases} \\
\text{(ii)} & h_i(P) & \begin{cases} 1, & i=0, \\ \nu-d, & 1\le i\le n, \end{cases} \\
\text{(iii)} & h_i(\Sigma(P)\backslash v) & \begin{cases} 1, & i=0, \\ \nu-d-1, & i=1, \\ \nu-r-1, & 2\le i\le d-1, \\ 0, & i=d, \end{cases} \\
\text{(iv)} & f_j(\mathrm{lk}_{\Sigma(P)}v) & \begin{cases} \binom{d-1}{j+1}+(r-d+1)\binom{d-1}{j}, & 0\le j\le d-3, \\ (r-d+1)(d-2)+2, & j=d-2, \end{cases} \\
\text{(v)} & f_j(P) & \begin{cases} \binom{d}{j+1}+(\nu-d)\binom{d}{j}, & 0\le j\le d-2, \\ (\nu-d)(d-1)+2, & j=d-1, \end{cases} \\
\text{(vi)} & f_j(\Sigma(P)\backslash v) & \begin{cases} \binom{d}{j+1}+(\nu-d-1)\binom{d-1}{j}+(\nu-r-1)\binom{d-1}{j-1}, & 0\le j\le d-2, \\ (\nu-r-1)(d-2)+\nu-d, & j=d-1, \end{cases} \\
\text{(vii)} & h_i(\mathrm{lk}_{\Sigma(P)}v) & \binom{r-d+i}{i}, \quad 0\le i\le m, \\
\text{(viii)} & h_i(P) & \binom{\nu-d+i-2}{i}+\binom{r-d+i-1}{i-1}, \quad 0\le i\le n, \\
\text{(ix)} & h_i(\Sigma(P)\backslash v) & \begin{cases} \binom{\nu-d+i-2}{i}, & 0\le i\le n, \\ \binom{\nu-i-2}{d-i}, & n+1\le i\le d-2, \\ \nu-r-1, & i=d-1, \\ 0, & i=d, \end{cases} \\
\text{(x)} & f_j(\mathrm{lk}_{\Sigma(P)}v) & f_j(C(r,d-1)), \quad 0\le j\le d-2, \\
\text{(xi)} & f_j(P) & f_j(C(\nu-1,d))+f_j(C(r+1,d))-f_j(C(r,d)), \quad 0\le j\le d-1, \\
\text{(xii)} & f_j(\Sigma(P)\backslash v) & \begin{cases} f_j(C(\nu-1,d)), & 0\le j\le d-3, \\ f_{d-2}(C(\nu-1,d))+d-r, & j=d-2, \\ f_{d-1}(C(\nu-1,d))+d-r-1, & j=d-1, \end{cases}
\end{array}
$$

where we formally define

$$
f_j(C(d,d))=\begin{cases} \binom{d}{j+1}, & 0\le j\le d-2, \\ 2, & j=d-1. \end{cases}
$$

Moreover, there exist polytope pairs (P_1, v_1), (P_2, v_2) and (P_3, v_3) each of type (d, ν, r) such that $h_i(\mathrm{lk}_{\Sigma(P_1)}v_1)$, $h_i(P_1)$, $h_i(\Sigma(P_1)\backslash v_1)$, $f_j(\mathrm{lk}_{\Sigma(P_1)}v_1)$, $f_j(P_1)$ and $f_j(\Sigma(P_1)\backslash v_1)$ are the values given in (i) through (vi), respectively; $h_i(\mathrm{lk}_{\Sigma(P_2)}v_2)$, $h_i(P_2)$, $f_j(\mathrm{lk}_{\Sigma(P_2)}v_2)$ and $f_j(P_2)$ are the values given in (vii), (viii), (x) and (xi), respectively; and $h_i(\Sigma(P_3)\backslash v_3)$ and $f_j(\Sigma(P_3)\backslash v_3)$ are the values given in (ix) and (xii), respectively.

PROOF. Define $h^{(1)}, h^{(2)} \in h(\mathscr{P}_s^{d-1})$ by

$$h_i^{(1)} = \begin{cases} 1, & i = 0 \text{ or } d-1, \\ r-d+1, & 1 \leq i \leq d-2, \end{cases}$$

$$h_i^{(2)} = \begin{cases} \binom{r-d+i}{i}, & 0 \leq i \leq m, \\ \binom{r-i-1}{d-i-1}, & m+1 \leq i \leq d-1. \end{cases}$$

As h ranges over all $h \in h(\mathscr{P}_s^{d-1})$ such that $h_1 = r-d+1$, we have as a consequence of McMullen's conditions 3.12 that

$$\begin{aligned} &\min h_i = r-d+1 = h_i^{(1)}, && 1 \leq i \leq d-2, \\ &\min (h_i - h_{i-1}) = 0 = h_i^{(1)} - h_{i-1}^{(1)}, && 2 \leq i \leq m, \\ &\max h_i = \binom{r-d+i}{i} = h_i^{(2)}, && 0 \leq i \leq m, \\ &\max (h_i - h_{i-1}) = 0 = h_i^{(1)} - h_{i-1}^{(1)}, && m+1 \leq i \leq d-2. \end{aligned}$$

Therefore, as (P, v) ranges over all polytope pairs of type (d, ν, r), Theorem 4.2 implies

$$\begin{aligned} \min h_i(P) &= \lambda_i^1(d, \nu, h^{(1)}), && 0 \leq i \leq d, \\ \min h_i(\Sigma(P)\backslash v) &= \lambda_i^2(d, \nu, h^{(1)}), && 0 \leq i \leq d, \\ \max h_i(P) &= \mu_i^1(d, \nu, h^{(2)}), && 0 \leq i \leq d, \\ \max h_i(\Sigma(P)\backslash v) &= \mu_i^2(d, \nu, h^{(1)}), && 0 \leq i \leq d. \end{aligned}$$

This establishes (i) through (iii) and (vii) through (ix).

The minima and maxima for the various f_j are determined by the facts that the f_j are non-negative linear combinations of the h_i, and that in each case the extremal values of the h_i are simultaneously achievable. In particular, let (P_1, v_1) be a polytope pair of type $(d, \nu, h^{(1)})$ achieving all of the values in (i) through (iii); (P_2, v_2) be a polytope pair of type $(d, \nu, h^{(2)})$ achieving all of the values in (vii) and (viii); and (P_3, v_3) be a polytope pair of type $(d, \nu, h^{(1)})$ achieving all of the values in (ix).

That (P_1, v_1) achieves the values in (iv) through (vi) is a straightforward computation. Recalling

$$h_i(C(\nu, d)) = \binom{\nu-d+i-1}{i}, \qquad 0 \leq i \leq [d/2],$$

and formally defining $h_i(C(d, d)) = \binom{i-1}{i}$, $0 \leq i \leq [d/2]$, allows us to confirm that (P_2, v_2) achieves the values in (x) and (xi), and (P_3, v_3) achieves the values in (xii), noting for (xi) that

$$\begin{aligned} h_i(P_2) &= \binom{v-d+i-2}{i} + \binom{r-d+i-1}{i-1} \\ &= \binom{v-d+i-2}{i} + \binom{r-d+i}{i} - \binom{r-d+i-1}{i} \\ &= h_i(C(v-1, d)) + h_i(C(r+1, d)) - h_i(C(r, d)), \qquad 0 \leq i \leq n. \end{aligned}$$

If S is a (not necessarily simplicial) $(d-1)$-dimensional *spherical complex* (as defined in [15, §4.1]) with v vertices, one of which is on exactly r edges, then in fact the value in 4.3(xi) provides a tight upper bound for $f_j(S)$ also. This is accomplished by triangulating S in an analogous manner to that of "pulling vertices" of a d-polytope [15, §2.5], and then invoking the extension of Theorem 4.2(iii) to pointed spheres and the Upper Bound Theorem for simplicial spheres [18].

Suppose (P, v) is a polytope pair of type (d, v, r). Let P^* be a simple d-polytope dual to P and F^* be the facet of P^* that corresponds to v under duality. Then all of the above f-vector results can be recast in a dual fashion for $f_j(F^*)$, $f_j(P^*)$ and $f_j(P^* \sim F^*)$. In particular, using $v+1$ instead of v, we have the following corollary.

COROLLARY 4.4. *Let $3 \leq d \leq r \leq v$. As P ranges over all simple d-polyhedra with v facets, exactly r of which are unbounded, then*

(i) $$\min f_j(P) = \begin{cases} (v-r)(d-2)+v-d+1, & j=0, \\ \binom{d}{j} + (v-d)\binom{d-1}{j} + (v-r)\binom{d-1}{j+1}, & 1 \leq j \leq d-1. \end{cases}$$

(ii) $$\max f_j(P) = \begin{cases} f_{d-1}(C(v, d)) + d - r - 1, & j=0, \\ f_{d-2}(C(v, d)) + d - r, & j=1, \\ f_{d-j-1}(C(v, d)), & 2 \leq j \leq d-1. \end{cases}$$

Parts (iv); (v); (vi) for $j = d-1$; (x); and (xii) for $j = d-1$ of Corollary 4.3, and parts (i) for $j=0$ and (ii) for $j=0$ of Corollary 4.4, were previously shown in a dual fashion by Klee [11]. Part (i) of 4.4 confirms a conjecture of Björner [5]. This paper was originally motivated by attempting to determine the values of 4.3(xi) for $j = d-1$. If one examines links of higher dimensional faces of polyhedral spheres, Theorem 4.2 can be considerably strengthened and Corollary 4.4 can be refined to take into account the dimension of the recession cone of P [13].

5. h-VECTORS OF POLYHEDRAL BALLS

Since the h-vectors of polyhedral spheres, that is, of simplicial polytopes, have been completely characterized, a reasonable question to ask at this point is whether the same might be accomplished for polyhedral balls. A number of necessary conditions for a vector $h = (h_0, h_1, \ldots, h_d)$ to be the h-vector of $\Sigma(P)\backslash v$ for some simplicial d-polytope P with vertex v have been encountered, the strongest being that of Corollary 3.14. We conclude this paper by conjecturing that, in fact, this condition is also sufficient. Namely:

CONJECTURE 5.1. *Let $h = (h_0, h_1, \ldots, h_d)$ be a $(d+1)$-vector of integers. Then $h = h(\Delta)$ for some polyhedral $(d-1)$-ball Δ if and only if*

$$(h_0(\Delta) - h_{d+k}(\Delta), h_1(\Delta) - h_{d+k-1}(\Delta), \ldots, h_m(\Delta) - h_{d+k-m}(\Delta))$$

is an O-sequence for all integer $0 \leq k \leq d+1$, where $m = [(d+k-1/2]$.

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